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First Digit Counting Compatibility for Niven Integer Powers

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Abstract.

It is claimed that the first digits of Niven integer powers follow a generalized Benford law with a specific parameter-free size-dependent exponent that converges asymptotically to the inverse power exponent. Numerical and other mathematical evidence, called first digit counting compatibility, is provided for this statement.

Keywords: First digit; Niven number; asymptotic counting function; probabilistic number theory; mean absolute deviation; probability weighted least squares.

Keywords: 11K36, 11N37, 11Y55, 62E20, 62F12.

1. Introduction

It is well-known that the first digits of many numerical data sets are not uniformly distributed. Newcomb [13] and Benford [1] observed that the first digits of many series of real numbers obey *Benford's law*

$$P^{B}(d)=\log_{10}(1+d)-\log_{10}(d), \quad d=1,2,\ldots,9. \tag{1.1}$$

The increasing knowledge about Benford's law and its applications has been collected in two recent books by Berger and Hill [2] and Miller [12]. In number theory, it is known that for any fixed power exponent $s\geq 1$, the first digits of some integer sequences, like integer powers and prime powers, follow asymptotically a Generalized Benford law (GB) with exponent $\alpha=s^{-1}\in(0,1]$ (e.g. Hürlimann [10]) such that

$$P_{\alpha}^{GB}(d)=\frac{(1+d)^{\alpha}-d^{\alpha}}{10^{\alpha}-1}, \quad d=1,2,\ldots,9. \tag{1.2}$$

Clearly, the limiting case $\alpha\to 0$, respectively $\alpha=1$, of (1.2), converges weakly to Benford's law, respectively the uniform distribution. It is expected that the first digits of many important integer power sequences follow a GB with size-dependent parameter. However, if asymptotically such an exponent exists, it may be exactly $\alpha=s^{-1}$, as advanced in [5] for square-free integer powers, but it may also differ from it. For example, Hürlimann [6,7] obtains numerical and another mathematical evidence, the so-called *first digit counting compatibility*, for the conjectural statement that the first digits of powers from perfect power numbers and powerful integer powers follow asymptotically a GB with parameter $\alpha=(2s)^{-1}$.

We study the first digits of powers from Niven numbers along the line of previous analysis. For this, the GB with unknown exponent α is fitted to samples of first digits from powers of Niven numbers using two goodness-of-fit measures, namely the mean absolute deviation measure, abbreviated MAD, and the probability weighted least square measure, abbreviated WLS. In Section 2, we determine the minimum MAD and WLS estimators of the GB for samples of first digits over finite ranges of powers up to $10^{s \cdot m}, m \geq 10$, $s \geq 1$ a fixed power exponent. Moreover, these minimum MAD and WLS estimators of the GB exponent α are compared to a size-dependent GB exponent of the form $\alpha(N,s) = s^{-1}\{1 - \ln(N)^{-1}\}$ over the finite range of Niven integer powers less than $N^s = 10^{5m \cdot s}, m = 2, \ldots, 6$, where $s = 1, 2, 3, 4, 5$ is a fixed power exponent. The absolute deviations of the size-dependent exponent $\alpha(N,s)$ to the minimum MAD and WLS estimators decrease with increasing sample size. Besides a precise measurement of the rate of convergence, the obtained numerical statistics motivate the asymptotic convergence of first digits of Niven integer powers to a GB with exponent s^{-1}. Section 3 relies on the criterion of first digit counting compatibility recently introduced in [10]. In general, this criterion permits to decide whether or not a given size-dependent GB that belongs to the first digits of some integer sequence is compatible with the asymptotic counting function of this sequence, if it exists. Theorem 3.1 shows the existence of a parameter-free size-dependent GB for the sequence of Niven integer powers that is first digit counting compatible with its known asymptotic counting function. Besides the numerical support stated above, this result provides further mathematical evidence for the assertion that the asymptotic distribution of the first digits of Niven integer powers follows a GB with exponent s^{-1}. We conclude with a brief outlook on related and future work.

2. Size-dependent generalized Benford law for powers of Niven numbers

A *Niven number* (or Harshad number) is a positive integer that is divisible by the sum of its (decimal) digits (sequence A005349 in Sloane's OEIS, URL: https://oeis.org/). Kennedy and Cooper [11] have shown that the set of Niven numbers is of zero density. De Koninck and Doyon [3] conjectured that the number of Niven numbers below x, denoted by $S(x)$, satisfies the asymptotic relationship

$$S(x) = c \cdot \frac{x}{\ln(x)} + o\left(\frac{x}{\ln(x)}\right), \quad x \to \infty, \quad c = \frac{14}{27} \cdot \ln(10), \tag{2.1}$$

a result proven in De Koninck et al. [4].

To investigate the optimal fit of a GB with unknown exponent α to first digit samples of Niven integer powers, it is necessary to specify goodness-of-fit (GoF) measures according to which optimality should hold. For this purpose, we use the following two GoF measures. Let $\{x_n\} \subset [1, \infty)$, $n \geq 1$, be an integer sequence, and let d_n be the (first) significant digit of x_n. The number of x_n's, $n = 1, \ldots, N$, with significant digit $d_n = d$ is denoted by $X_N(d)$. The *MAD measure* or *mean absolute deviation* measure for the GB is defined to be

$$MAD_N(\alpha) = \frac{1}{9} \cdot \sum_{d=1}^{9} \left| P_\alpha^{GB}(d) - \tfrac{X_N(d)}{N} \right|. \tag{2.2}$$

The *WLS measure* (chi-square divided by sample size) for the GB is defined by

$$WLS_N(\alpha) = \frac{1}{N} \cdot \sum_{d=1}^{9} \frac{(P_\alpha^{GB}(d) - \frac{X_N(d)}{N})^2}{P_\alpha^{GB}(d)}. \tag{2.3}$$

These statistical measures have been widely used in scientific research to assess conformity to Benford's law and some of its extensions (for the latter, see [8,9]). Consider now the sequence of integer powers $\{n_{Nv}^s\}, n_{Nv}^s < 10^{s \cdot m}$, for a fixed exponent $s = 1, 2, 3, \ldots$, and arbitrary Niven numbers n_{Nv} below $10^m, m \geq 10$. Denote by $I_k^s(d)$ the number of Niven integer powers below $10^k, k \geq 1$, with first digit d. This number is defined recursively by the relationship

$$I_{k+1}^s(d) = S(\sqrt[s]{(d+1) \cdot 10^k}) - S(\sqrt[s]{d \cdot 10^k}) + I_k^s(d), \quad k = 1, 2, \ldots, \tag{2.4}$$

with $S(x)$ the counting function of Niven numbers. Although exact values of $S(x)$ can be obtained using computers (e.g. De Koninck and Doyon [3]), a simple efficient algorithm to evaluate this counting function does not seem to be known so far. At the cost of some loss in accuracy, we avoid these computational difficulties by using appropriate approximation formulas for $S(x)$. Since we are mostly interested in the asymptotic behaviour of the first digits, we replace the exact value of the counting function by the asymptotic formula $S_{as}(x) = c \cdot x / \ln(x)$.

In general, with $N = S(10^m)$ one has $X_N(d) = I^s_{s \cdot m}(d)$ in (2.2)-(2.3). Based on (2.1) a list of approximate values for $I^s_{3m \cdot s}(d), m = 2, \dots, 6, s = 1, 2, 3, 4, 5$, together with approximate sample sizes $N = S_{as}(10^{3m})$, is provided in Table 3 of the Appendix. Based on the above, we have determined the so-called minimum MAD and minimum WLS estimators for the GB with unknown parameter α. Together with their GoF measures, these optimal estimators are reported in Table 1 below. Note that the minimum WLS is a critical point of the equation

$$\frac{\partial}{\partial \alpha} WLS_N(\alpha) = \sum_{d=1}^{9} \frac{\partial P_\alpha^{GB}(d)}{\partial \alpha} \cdot \frac{P_\alpha^{GB}(d)^2 - (\frac{X_N(d)}{N})^2}{P_\alpha^{GB}(d)^2} = 0 \text{, with}$$

$$\frac{\partial P_\alpha^{GB}(d)}{\partial \alpha} = \frac{(1+d)^\alpha \{\ln(\frac{1+d}{10})10^\alpha - \ln(1+d)\} - d^\alpha \{\ln(\frac{d}{10})10^\alpha - \ln(d)\}}{(10^\alpha - 1)^2}, \quad d = 1, 2, \dots, 9. \qquad (2.5)$$

For comparison, the MAD and WLS measures for a size-dependent GB exponent of the form

$$\alpha(N,s) = s^{-1}\{1 - \ln(N)^{-1}\} \qquad (2.6)$$

are also listed. Given finite samples of Niven integer powers $\{n^s_{Nv}\}, n^s_{Nv} < N^s$, denote the corresponding minimum MAD and WLS estimators by $\alpha_{MAD}(N,s)$ respectively $\alpha_{WLS}(N,s)$, and consider their absolute distances to the LL estimator (2.6), which correspond to the columns "Δ to LL estimate" in Table 1. Numerical results are listed for Niven integer powers over the finite intervals $[1, N^s], N = 10^{5m}, m = 2, \dots, 6, s = 1, 2, 3, 4, 5$. The MAD (respectively WLS) measures are given in units of 10^{-6} (respectively $10^{-(5m+10)}$) while the absolute deviations, which correspond to the columns "Δ to LL estimate" in Table 1, are given in units of $s \cdot 10^{-5}$. The minimum MAD and WLS measures, as well as their distances to the LL estimator, decrease with increasing sample size, as should be for convergence to a GB with exponent s^{-1}.

Table 1. GB fit for first digits of Niven integer powers: MAD vs WLS criterion

s=1	parameters		Δ to LL estimate		MAD GoF measures			WLS GoF measures		
m=	WLS	MAD	WLS	MAD	LL	WLS	**MAD**	LL	**WLS**	MAD
2	0.95618390	0.95650050	38.67	7.005	30.324	31.283	**29.282**	29148	**19226**	25879
3	0.97087196	0.97100624	17.51	4.079	13.431	13.761	**12.851**	8633.1	**5601.7**	7385.0
4	0.97818593	0.97825995	9.934	2.532	7.5393	7.6988	**7.1824**	3639.6	**2342.4**	3062.5
5	0.98256432	0.98261074	6.390	1.748	4.8185	4.9109	**4.5796**	1862.4	**1192.8**	1546.2
6	0.98547899	0.98551077	4.453	1.275	3.3446	3.4027	**3.1724**	1077.3	**687.72**	886.13

s=2	parameters		Δ to LL estimate		MAD GoF measures			WLS GoF measures		
m=	WLS	MAD	WLS	MAD	LL	WLS	**MAD**	LL	**WLS**	MAD
2	0.47823725	0.47821742	9.605	13.57	8.5964	8.4550	**8.3967**	1485.9	**1306.0**	1336.6
3	0.48550156	0.48549244	4.392	6.216	3.8097	3.7509	**3.7265**	441.99	**385.67**	395.39
4	0.48913011	0.48912486	2.505	3.555	2.1399	2.1080	**2.0946**	186.84	**162.43**	166.72
5	0.49130603	0.49130266	1.616	2.290	1.3683	1.3483	**1.3400**	95.771	**83.074**	85.278
6	0.49275612	0.49275370	1.129	1.612	0.9496	0.9360	**0.9301**	55.465	**48.040**	49.401

s=3	parameters		Δ to LL estimate		MAD GoF measures			WLS GoF measures		
m=	WLS	MAD	WLS	MAD	LL	WLS	**MAD**	LL	**WLS**	MAD
2	0.31884389	0.31884058	3.888	4.880	3.8870	3.7727	**3.7447**	273.63	**260.12**	261.00
3	0.32367639	0.32367476	1.787	2.276	1.7280	1.6765	**1.6637**	81.329	**77.052**	77.372
4	0.32609169	0.32609080	1.022	1.289	0.9720	0.9429	**0.9353**	34.355	**32.491**	32.618
5	0.32754054	0.32753993	0.6603	0.8416	0.6221	0.6034	**0.5985**	17.603	**16.630**	16.704
6	0.32850630	0.32850588	0.4615	0.5875	0.4320	0.4190	**0.4156**	10.192	**9.6219**	9.6644

s=4	parameters		Δ to LL estimate		MAD GoF measures			WLS GoF measures		
m=	WLS	MAD	WLS	MAD	LL	WLS	**MAD**	LL	**WLS**	MAD
2	0.23913777	0.23911574	1.946	10.76	2.1741	2.1169	**2.0906**	84.252	**82.329**	121.77
3	0.24275952	0.24274984	0.8959	4.768	0.9675	0.9415	**0.9302**	25.030	**24.419**	35.833
4	0.24457004	0.24456457	0.5136	2.701	0.5444	0.5296	**0.5234**	10.568	**10.301**	15.158
5	0.24565622	0.24565276	0.3324	1.717	0.3485	0.3389	**0.3351**	5.4136	**5.2735**	7.7035
6	0.24638030	0.24637801	0.2325	1.146	0.2421	0.2354	**0.2329**	3.1339	**3.0516**	4.3223

s=5	parameters		Δ to LL estimate		MAD GoF measures			WLS GoF measures		
m=	WLS	MAD	WLS	MAD	LL	WLS	**MAD**	LL	**WLS**	MAD
2	0.19131195	0.19129798	1.079	8.064	1.3831	1.3554	**1.3163**	34.120	**33.740**	49.669
3	0.19420841	0.19420232	0.5002	3.542	0.6145	0.6023	**0.5859**	10.121	**9.998**	14.528
4	0.19565648	0.19565310	0.2876	1.977	0.3458	0.3387	**0.3298**	4.2722	**4.2182**	6.0803
5	0.19652527	0.19652304	0.1864	1.302	0.2214	0.2168	**0.2109**	2.1881	**2.1597**	3.1755
6	0.19710444	0.19710292	0.1306	0.8897	0.1538	0.1505	**0.1466**	1.2665	**1.2499**	1.8139

3. First digit counting compatibility for Niven integer powers

Table 1 provides a strong numerical support for the approximation $\frac{I_{5m\cdot s}^{s}(d)}{\pi(10^{5m})} \approx P_{\alpha(10^{5m},s)}^{GB}(d)$, whose precision increases by growing value of m. Since $\alpha(10^m, s) \to s^{-1}$ $(m \to \infty)$ this approximation suggests the asymptotic convergence $\frac{I_{s\cdot m}^{s}(d)}{\pi(10^m)} \to P_{s^{-1}}^{GB}(d)$ $(m \to \infty)$, which tells us that the relative density of the first digits of Niven integer powers converges asymptotically to a GB with exponent s^{-1}. Although we are unable to prove this statement rigorously, it is possible to support its validity through application of the first digit counting compatibility criterion introduced in [10].

Recall its definition. Let $\{x_n\}, n \geq 1$, be an arbitrary integer sequence, and suppose that the asymptotic counting function $Q(N)$ as $N \to \infty$ of this sequence exists. Further, let $\alpha(N) \in (0,1]$ be a size-dependent exponent such that the sequence of numbers generated by the power-law density $x^{-\alpha(N)}$, has a GB first digit distribution $P_{1-\alpha(N)}^{GB}(d)$ with exponent $1-\alpha(N)$.

Definition 3.1. The generalized Benford law $P_{1-\alpha(N)}^{GB}(d)$ is *counting compatible* with the counting function $Q(N)$ if there exists a constant $c(N)$ such that the generalized Benford counting function defined by $c(N) \cdot \int_2^N x^{-\alpha(N)} dx$ is asymptotically equivalent to $Q(N)$.

Let us apply this criterion to the sequence of Niven integer powers. Starting point is the asymptotic counting function (2.1) for Niven numbers, which give their total number in the inteval $[1, N]$, denoted by $Q(N)$. It is given by

$$Q(N) = c \cdot N / \ln N, \quad (N \to \infty), \quad c = \frac{14}{27} \cdot \ln(10). \tag{3.1}$$

Similarly, for any fixed positive integer $s \geq 1$, the number of Niven integer powers n_{Nv}^s in the interval $[1, N^s]$, denoted by $Q_s(N^s)$, follows the same asymptotic distribution

$$Q_s(N^s) = c \cdot N / \ln N, \quad (N \to \infty). \tag{3.2}$$

This follows from the fact that $n_{Nv}^s < N^s$ if, and only if, one has $n_{Nv} < N$. In the notation of Definition 3.1, consider the following slightly modified parametric GB size-dependent exponent that corresponds to (2.6), namely

$$\tilde{\alpha}(s, N, a) = \frac{s - 1 + \tilde{\beta}(N, a)}{s}, \quad \tilde{\beta}(N, a) = \frac{a}{\ln(N)}. \tag{3.3}$$

Theorem 3.1 (*Counting compatibility of the GB for Niven integer powers*). For any fixed positive integer $s \geq 1$ and any $m \geq 1$, set

$$\alpha(s, m, a) = 1 - \tilde{\alpha}(s, 10^m, a) = \frac{1}{s}\left(1 - a \cdot \ln(10^m)^{-1}\right). \tag{3.4}$$

Then, the generalized Benford law $P_{\alpha(s,m,a)}^{GB}(d)$, $d = 1, \ldots, 9$, is counting compatible with the Niven power counting function (3.2) if, and only if, the parameter $a = 1$. More precisely, the choice of the constant

$$c(N, s, a) = \frac{c \cdot a}{s \cdot \ln N} \tag{3.5}$$

implies that the generalized Benford counting function $L_s(N^s) = c(N, s, a) \cdot \int_2^{N^s} x^{-\tilde{\alpha}(s,N,a)} dx$ is asymptotically equivalent to $Q_s(N^s) \sim c \cdot N / \ln N \ (N \to \infty)$ if, and only if, one has $a = 1$.

Proof. Counting compatibility holds provided the following limiting condition holds:

$$\lim_{N \to \infty} \frac{L_s(N^s)}{c \cdot N / \ln(N)} = 1. \tag{3.6}$$

Using (3.4) one obtains the equivalent asymptotic formula

$$L_s(N^s) \sim \frac{c \cdot \tilde{\beta}(N, a)}{s \cdot (1 - \tilde{\beta}(s, N, a))} N^{s \cdot (1 - \tilde{\alpha}(s, N, a))} = \frac{c \cdot \tilde{\beta}(N, a)}{1 - \tilde{\beta}(N, a)} N^{1 - \tilde{\beta}(N, a)} = \frac{c \cdot a \cdot N}{\ln(N) - a} \exp\left(\frac{-a}{\ln(N)}\right). \tag{3.7}$$

Clearly, the factor

$$f_N(a) = \frac{L_s(N^s)}{c \cdot N / \ln(N)} = \frac{a \cdot \ln(N)}{\ln(N) - a} \exp\left(\frac{-a}{\ln(N)}\right)$$

converges to 1 as $N \to \infty$ exactly when $a = 1$, and in this case counting compatibility holds. Moreover, the form (3.4) of the GB exponent in Definition 3.1 follows by setting $N = 10^s$ in Equation (3.3). The result is shown. ◊

Table 2 compares the new counting function $L_s(N^s) = L(N) = \frac{c \cdot N}{\ln(N) - 1} \exp\left(\frac{-1}{\ln(N)}\right)$ in (3.7) with $Q(N) = c \cdot N / \ln N$. One observes that $L(N)$ converges to $Q(N)$ from above.

Table 2. Comparison of Niven number counting functions for $N = 10^{3m}$

m	$Q(N)$	$L(N)$	$L(N)/Q(N)$
2	86'419	86'658	1.002766
3	57'613'168	57'682'526	1.001204
4	43'209'876'543	43'238'886'350	1.000671
5	34'567'901'234'567	34'582'678'959'674	1.000427
6	28'806'584'362'139'900	28'815'107'637'051'900	1.000296

To conclude, a brief outlook might stimulate further work on Benford's law and its extensions. Departures from Benford's law occur quite frequently within mathematics and in almost all related scientific disciplines. It is known that the first digits of integer powers and prime powers follow asymptotically a GB with inverse power exponent (e.g. [10], Theorem 5.1 and Remarks 5.1). For other integer sequences (with an asymptotic counting function) like square-free integer powers, powers of perfect powers, powerful integer powers, Niven integer powers, etc., the observed discrepancies can be explained in a non-trivial way. For them, there is strong support for the conjecture that the first significant digits of these sequences obey a generalized Benford law with size dependent parameter proportional to the inverse of a multiple of the power exponent. It seems that first digit counting compatibility is quite close to a proof of such a statement. Does counting compatibility always predict the correct asymptotic generalized Benford law? Although this question remains unsolved, counting compatibility clearly contains more information than the exact asymptotic relative density of first digits because it yields the analytical form of the size-dependent GB exponent over finite intervals.

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Appendix: Tables of first digits for powers of Niven numbers

Based on the recursive relation (2.4), the calculation of $I^{s}_{3m\cdot s}(d)$, $m=2,\ldots,6$, is straightforward. These numbers are listed in Table 3. The entry $s\to\infty$ corresponds to the limiting Benford law as the power goes to infinity.

Table 3. First digit distribution of Niven integer powers up to $10^{3m\cdot s}$, $m=2,\ldots,6$, $s=1,2,3,4,5$

s=1 / 1st digit	86'423	57'613'172	43'209'876'547	34'567'901'234'571	28'806'584'362'139'900
1	10'472	6'779'683	5'011'554'170	3'974'694'129'663	3'293'255'132'058'320
2	10'055	6'602'829	4'914'280'705	3'913'272'980'867	3'250'983'735'888'720
3	9'803	6'493'002	4'853'175'002	3'874'426'260'844	3'224'128'339'786'650
4	9'626	6'413'711	4'808'727'865	3'846'043'884'240	3'204'449'038'971'690
5	9'485	6'351'904	4'773'891'312	3'823'724'385'684	3'188'939'168'952'240
6	9'373	6'301'411	4'745'305'397	3'805'360'916'649	3'176'155'682'863'420
7	9'280	6'258'821	4'721'105'324	3'789'780'514'983	3'165'293'533'720'000
8	9'200	6'222'058	4'700'149'209	3'776'263'105'796	3'155'857'681'075'460
9	9'129	6'189'753	4'681'687'563	3'764'335'055'845	3'147'522'048'823'430

s=2 / 1st digit	86'419	57'613'168	43'209'876'543	34'567'901'234'567	28'806'584'362'139'900
1	17'170	11'307'108	8'429'049'247	6'718'766'521'479	5'585'425'540'013'590
2	12'912	8'562'652	6'404'511'652	5'115'181'531'522	4'257'956'262'560'630
3	10'751	7'158'705	5'365'646'947	4'290'833'884'900	3'574'738'604'306'270
4	9'384	6'268'419	4'705'552'318	3'766'424'919'010	3'139'771'332'038'690
5	8'425	5'639'688	4'238'686'758	3'395'197'618'572	2'831'679'607'929'200
6	7'696	5'165'482	3'886'144'956	3'114'675'801'948	2'598'757'676'443'080
7	7'132	4'791'523	3'607'847'191	2'893'099'708'132	2'414'707'240'645'460
8	6'667	4'486'922	3'380'976'801	2'712'377'409'194	2'264'541'126'007'390
9	6'282	4'232'669	3'191'460'673	2'561'343'839'810	2'139'006'972'195'620

s=3 / 1st digit	86'418	57'613'167	43'209'876'542	34'567'901'234'566	28'806'584'362'139'800
1	19'901	13'168'782	9'839'391'301	7'853'716'254'021	6'534'896'511'501'080
2	13'780	9'156'716	6'856'856'647	5'480'335'974'986	4'564'077'867'770'210
3	10'875	7'249'237	5'436'043'543	4'348'381'446'857	3'623'389'408'830'100
4	9'129	6'096'728	4'576'472'003	3'663'040'665'306	3'053'561'434'256'060
5	7'940	5'312'022	3'990'652'499	3'195'695'348'616	2'664'838'084'162'500
6	7'073	4'737'384	3'561'329'171	2'853'038'942'506	2'379'739'799'571'740
7	6'406	4'295'312	3'230'831'906	2'589'156'952'226	2'160'127'425'597'640
8	5'876	3'942'855	2'967'187'072	2'378'582'769'496	1'984'841'148'408'900
9	5'438	3'654'131	2'751'112'400	2'205'952'880'552	1'841'112'682'041'620

s=4 / 1st digit	86'417	57'613'166	43'209'876'541	34'567'901'234'565	28'806'584'362'139'900
1	21'357	14'160'005	10'591'066'699	8'458'993'778'275	7'041'472'281'607'390
2	14'179	9'432'085	7'066'508'309	5'649'566'181'170	4'705'944'208'013'140
3	10'899	7'265'944	5'449'328'499	4'359'383'992'607	3'632'770'543'025'310
4	8'968	5'988'590	4'494'779'474	3'597'413'428'411	2'998'723'746'476'980
5	7'678	5'134'696	3'856'221'140	3'087'465'138'196	2'574'267'817'364'360
6	6'754	4'518'592	3'395'216'573	2'719'183'642'431	2'267'658'380'016'620
7	6'050	4'050'464	3'044'772'157	2'439'144'606'704	2'034'469'837'088'980
8	5'492	3'681'217	2'768'234'943	2'218'109'963'634	1'850'383'981'389'320
9	5'040	3'381'573	2'543'748'747	2'038'640'503'137	1'700'893'567'157'820

s=5 / 1st digit	86'416	57'613'165	43'209'876'540	34'567'901'234'564	28'806'584'362'140'000
1	22'252	14'772'680	11'055'889'762	8'833'392'274'231	7'354'878'047'091'310
2	14'407	9'589'061	7'185'992'790	5'745'999'259'485	4'786'776'447'907'640
3	10'900	7'266'575	5'450'086'342	4'360'132'802'830	3'633'475'761'473'400
4	8'861	5'916'915	4'440'537'121	3'553'790'350'178	2'962'245'384'496'350
5	7'520	5'024'590	3'772'693'786	3'020'191'925'266	2'517'956'963'626'950
6	6'558	4'386'448	3'294'872'770	2'638'314'440'795	2'199'937'562'796'130
7	5'840	3'905'196	2'934'381'664	2'350'141'702'087	1'959'916'858'605'540
8	5'269	3'528'021	2'651'765'601	2'124'176'906'830	1'771'684'787'033'600
9	4'809	3'223'679	2'423'656'704	1'941'761'572'862	1'619'712'549'109'050